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TARGET SPACE DUALITY IN ORBIFOLDS WITH CONTINUOUS AND DISCRETE WILSON LINES.

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ABSTRACT

Duality symmetry is studied for heterotic string orbifold compactifications in the presence of a general background which in addition to the metric and antisymmetric tensor fields contains both discrete and continuous Wilson lines.

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Orbifold compactified heterotic string theories [1, 2, 3] provide phenomenologically promising string compactified backgrounds as they give rise to semi-realistic four dimensional quantum field theories [2, 6]. The orbifold models are characterized by a set of continuous parameters referred to as moduli. These moduli are the marginal deformations of the underlying conformal field theory of the orbifold. They appear in the space-time supersymmetric four dimensional Lagrangian with a flat potential to all orders in perturbation theory. The moduli space of the orbifold compactification is a subspace of that of the toroidal compactification obtained by demanding that the twist action on the underlying Narain lattice is an automorphism. The coset structure of the moduli space for the $\mathbf{Z}_3$ orbifold, in the absence of Wilson lines, has been given in [7] and for general $\mathbf{Z}_N$ orbifolds in [8]. However, in [8], the symmetries of the conformal field theory are employed to determine the Kahler potential which in turn fixes the form of the coset structure of the moduli space. In heterotic string theory, the moduli space of the orbifold is enlarged when the gauge twist in the $E_8 \times E_8$ root lattice is realized by a rotation. The extra moduli are continuous Wilson lines [3, 4, 5]. More recently, the coset structure of moduli space of the untwisted moduli including continuous Wilson lines has been identified in [9]. Locally, the untwisted moduli space is determined by the eigenvalues of the twist.

However, string theories have a novel discrete symmetry, the so-called target space duality [11-19] (see also [23] and references therein), consisting of discrete reparametrizations of the background fields (moduli) leaving the underlying conformal field theory invariant. Therefore, the physical moduli space has the form of an orbifold given as the quotient of the moduli space by a discrete symmetry group.

In two-dimensional $\mathbf{Z}_2$-orbifold compactification, or in an eigenspace of a six-dimensional orbifold where the twist has an eigenvalue -1 and in which the eigenspace lies entirely in a two-dimensional sublattice of the orbifold six-dimensional lattice, the duality symmetry is given by $SL(2,Z)_T \times SL(2,Z)_U$, where T and U are the complex moduli parametrizing the complex plane with the eigenvalue -1 [12].

This is the symmetry of the moduli-dependent threshold corrections to the gauge coupling constants [20-22] in this case. However if the complex plane does not lie entirely in a two-dimensional sublattice then the duality group is broken down to a subgroup of $SL(2,Z)_T \times SL(2,Z)_U$ [24-26]. Note that if the eigenvalue of the twist is different from -1 the U modulus is frozen to a constant phase factor and the duality symmetry is given by the T-duality. Also it is known that the presence of discrete Wilson lines break the duality group [10,19]. In [10], the duality symmetry of the moduli-dependent threshold corrections to the gauge coupling constants in the presence of discrete Wilson lines background was determined. Here we are interested in studying the duality symmetry of orbifold compactified heterotic string theory in the presence of a general background which contains both discrete and continuous Wilson lines. Of particular interest are the symmetries of the twisted sectors contributing to the threshold corrections of the gauge coupling constants.* Some consequences of discrete Wilson line contributions to threshold corrections has been recently discussed in [27] .

Consider a ten-dimensional $E_8 \times E_8$ heterotic string compactified on a d-dimensional torus [28] defined as a quotient of $\mathbf{R}^d$ with respect to a lattice Λ defined by

$$\Lambda = \Big\{ \sum_{i=1}^{d} a^i e_i, \qquad a^i \in Z \Big\}. \tag{1}$$

The background is described by the metric $G_{ij} = e_i.e_j$, an antisymmetric field B_{ij} and Wilson lines W_i^I, where the index I refer to the gauge degrees of freedom of the $E_8 \times E_8$ lattice. The left and right momenta for the compactified string coordinates are given by

$$\begin{aligned}\mathbf{P}_L =&\Big(\frac{\mathbf{m}}{2} + (\mathbf{G} - \mathbf{B} - \frac{1}{4}\mathbf{W}^t\mathbf{C}\mathbf{W})\mathbf{n} - \frac{1}{2}\mathbf{W}^t\mathbf{C}\mathbf{l},\ \mathbf{l} + \mathbf{W}\mathbf{n}\Big) = \Big(\mathbf{p}_L,\ \tilde{\mathbf{p}}_L\Big),\\ \mathbf{P}_R =&\Big(\frac{\mathbf{p}}{2} - (\mathbf{G} + \mathbf{B} + \frac{1}{4}\mathbf{W}^t\mathbf{C}\mathbf{W})\mathbf{n} - \frac{1}{2}\mathbf{W}^t\mathbf{C}\mathbf{l},\ \mathbf{0}\Big) = \Big(\mathbf{p}_R,\ \mathbf{0}\Big),\end{aligned} \tag{2}$$

$*$ Recent progress has been made concerning the calculation of such threshold corrections in the presence of continuous Wilson lines [30]

where $\mathbf{n}$ and $\mathbf{m}$ the windings and momenta respectively, are d-dimensional integer valued vectors, $\mathbf{l}$ is a 16-dimensional vector representing the internal gauge quantum numbers and $\mathbf{C}$ is the Cartan metric of the self-dual Euclidean root lattice of $E_8 \times E_8$.

The left and right momenta of the compactified string coordinates have the contribution H and S to the scaling dimension and spin of the vertex operators of the underlying conformal field theory given by

$$\begin{aligned} H =& \frac{1}{2}\Big(\mathbf{p}_L^t\mathbf{G}^{-1}\mathbf{p}_L + \tilde{\mathbf{p}}_L^t\mathbf{C}\tilde{\mathbf{p}}_L + \mathbf{p}_R^t\mathbf{G}^{-1}\mathbf{p}_R\Big), \\ S =& \frac{1}{2}\Big(\mathbf{p}_L^t\mathbf{G}^{-1}\mathbf{p}_L + \tilde{\mathbf{p}}_L^t\mathbf{C}\tilde{\mathbf{p}}_L - \mathbf{p}_R^t\mathbf{G}^{-1}\mathbf{p}_R\Big). \end{aligned} \tag{3}$$

One way to study the duality transformations of the spectrum is to write H and S in the following quadratic forms [29]

$$H = \frac{1}{2}\mathbf{u}^t\boldsymbol{\Xi}\mathbf{u}, \qquad S = \frac{1}{2}\mathbf{u}^t\eta\mathbf{u}, \tag{4}$$

where

$$\begin{aligned} &\mathbf{u} = \begin{pmatrix} \mathbf{n} \\ \mathbf{m} \\ \mathbf{l} \end{pmatrix}, \quad \eta = \begin{pmatrix} \mathbf{0} & \mathbf{1}_d & \mathbf{0} \\ \mathbf{1}_d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} \end{pmatrix}, \\ &\boldsymbol{\Xi} = \begin{pmatrix} \frac{1}{2}\mathbf{D}\mathbf{G}^{-1}\mathbf{D}^t & \mathbf{1}_d + \frac{1}{2}\mathbf{D}\mathbf{G}^{-1} & -\frac{1}{2}\mathbf{D}\mathbf{G}^{-1}\mathbf{W}^t\mathbf{C} \\ \mathbf{1}_d + \frac{1}{2}\mathbf{G}^{-1}\mathbf{D}^t & \frac{1}{2}\mathbf{G}^{-1} & -\frac{1}{2}\mathbf{G}^{-1}\mathbf{W}^t\mathbf{C} \\ -\frac{1}{2}\mathbf{C}\mathbf{W}\mathbf{G}^{-1}\mathbf{D}^t & -\frac{1}{2}\mathbf{C}\mathbf{W}\mathbf{G}^{-1} & \mathbf{C} + \frac{1}{2}\mathbf{C}\mathbf{W}\mathbf{G}^{-1}\mathbf{W}^t\mathbf{C} \end{pmatrix} \end{aligned} \tag{5}$$

with $\mathbf{D} = 2\Big(\mathbf{B} - \mathbf{G} - \frac{1}{4}\mathbf{W}^t\mathbf{C}\mathbf{W}\Big)$ and $\mathbf{1}_d$ is the $d \times d$ identity matrix.

The discrete target space duality symmetries are then defined to be all integer-valued linear transformations of the quantum numbers $\mathbf{n}, \mathbf{m}$ and $\mathbf{l}$ leaving the spectrum invariant. Denote these linear transformations by $\boldsymbol{\Omega}$ and define their

action on the quantum numbers as

$$\mathbf{\Omega} : \mathbf{u} \longrightarrow \mathbf{\Omega}^{-1}\mathbf{u}. \tag{6}$$

In order for these discrete transformation to preserve S, the transformation matrix $\mathbf{\Omega}$ should satisfy the condition:

$$\mathbf{\Omega}^t \eta \mathbf{\Omega} = \eta. \tag{7}$$

This means that $\mathbf{\Omega}$ is an element of $O(d+16, d; Z)$. Moreover, requiring the invariance of H under the duality transformation induces a transformation on the moduli. Such a transformation defines the action of the duality group and is given by

$$\mathbf{\Omega} : \mathbf{\Xi} \longrightarrow \mathbf{\Omega}^t \mathbf{\Xi} \mathbf{\Omega}. \tag{8}$$

The above analysis can be generalized to the orbifold case including both discrete and continuous Wilson lines in the following manner. Consider the case where the twist acts on the orbifold and the $E_8 \times E_8$ lattice vectors. To define the action of the twist on the quantum numbers we demand that under the action of the twist both $\mathbf{p}_L$ and $\mathbf{p}_R$ are rotated by $\mathbf{Q}^* = \mathbf{Q}^{t^{-1}}$ and $\tilde{\mathbf{p}}_L$ is rotated by $\mathbf{M}$, where $\mathbf{Q}$ is a $d \times d$ matrix defining the action of the twist on the d-dimensional orbifold lattice, $\mathbf{M}$ is the action of the twist on the $E_8 \times E_8$ lattice satisfying $\mathbf{M}^t\mathbf{C}\mathbf{M} = \mathbf{C}$. Using the fact that the winding numbers transform as $\mathbf{n} \longrightarrow \mathbf{Q}\mathbf{n}$ then for $\tilde{\mathbf{p}}_L$ to be rotated by $\mathbf{M}$ and such that the quantum numbers $\mathbf{l}$ are transformed as integers, the Wilson line must satisfy

$$\mathbf{M}\mathbf{W} - \mathbf{W}\mathbf{Q} = \mathbf{V} \in Z. \tag{9}$$

If we also demand that the background fields $\mathbf{G}$ and $\mathbf{B}$ satisfy $\mathbf{Q}^t\mathbf{G}\mathbf{Q} = \mathbf{G}$ and $\mathbf{Q}^t\mathbf{B}\mathbf{Q} = \mathbf{B}$, then $\mathbf{p}_L$ and $\mathbf{p}_R$ are rotated by $\mathbf{Q}^*$, with the appropriate transformation of the momenta $\mathbf{m}$, provided that $\mathbf{V}$ does not have entries in the rotated

directions of the $E_8 \times E_8$ lattice. Moreover if one decomposes the Wilson line as $\mathbf{W} = \mathbf{A} + \mathbf{a}$ where $\mathbf{a}$ is the unrotated piece under the action of $\mathbf{M}$, and $\mathbf{A}$ is perpendicular to $\mathbf{a}$ in the sense that $\mathbf{A}^t\mathbf{C}\mathbf{a} = 0$, then it can be shown [5] that $\mathbf{A}$ takes continuous values while $\mathbf{a}$ is discrete. In this case the condition (9) gives

$$\begin{aligned} \mathbf{M}\mathbf{A} - \mathbf{A}\mathbf{Q} &= \mathbf{0} \\ \mathbf{a} - \mathbf{a}\mathbf{Q} &= \mathbf{V} \in Z, \end{aligned} \tag{10}$$

and the action of the twist on the quantum numbers can be represented as

$$\mathbf{u} \longrightarrow \mathcal{R}\mathbf{u}; \quad \mathcal{R} = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \alpha & \mathbf{Q}^* & \gamma \\ \mathbf{a}(\mathbf{1}_d - \mathbf{Q}) & \mathbf{0} & \mathbf{M} \end{pmatrix}, \tag{11}$$

where α, γ are the matrices

$$\begin{aligned} \alpha &= \frac{1}{2}\mathbf{a}^t\mathbf{C}\mathbf{a}(\mathbf{1}_d - \mathbf{Q}) + \frac{1}{2}(\mathbf{1}_d - \mathbf{Q}^*)\mathbf{a}^t\mathbf{C}\mathbf{a} \in Z, \\ \gamma &= (\mathbf{1}_d - \mathbf{Q}^*)\mathbf{a}^t\mathbf{C} \in Z. \end{aligned} \tag{12}$$

Note that only those components of $\mathbf{l}$ that are not rotated by $\mathbf{M}$ get shifted by $\mathbf{a}(\mathbf{1}_d - \mathbf{Q})\mathbf{n}$.

In order to study the duality symmetries in the presence of both discrete and continuous Wilson lines, we employ the method used in [10] and define the new basis $\mathbf{u}'$

$$\mathbf{u}' = \mathbf{T}\mathbf{u} = \begin{pmatrix} \mathbf{1}_d & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{a}^t\mathbf{C}\mathbf{a} & \mathbf{1}_d & -\mathbf{a}^t\mathbf{C} \\ \mathbf{a} & \mathbf{0} & \mathbf{1}_{16} \end{pmatrix} \mathbf{u} \tag{13}$$

In this basis the twist matrix is diagonal

$$\mathcal{R}' = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{pmatrix}, \tag{14}$$

Due to the fact that $\mathbf{A}^t\mathbf{C}\mathbf{a} = \mathbf{0}$, the scaling dimension and spin H and S take the

form

$$H = \frac{1}{2}\mathbf{u}'^t \mathbf{\Xi}' \mathbf{u}', \qquad S = \frac{1}{2}\mathbf{u}'^t \eta \mathbf{u}' \tag{15}$$

where

$$\mathbf{\Xi}' = \begin{pmatrix} \frac{1}{2}\mathbf{D}'\mathbf{G}^{-1}\mathbf{D}'^t & \mathbf{1}_d + \frac{1}{2}\mathbf{D}'\mathbf{G}^{-1} & -\frac{1}{2}\mathbf{D}'\mathbf{G}^{-1}\mathbf{A}^t\mathbf{C} \\ \mathbf{1}_d + \frac{1}{2}\mathbf{G}^{-1}\mathbf{D}'^t & \frac{1}{2}\mathbf{G}^{-1} & -\frac{1}{2}\mathbf{G}^{-1}\mathbf{A}^t\mathbf{C} \\ -\frac{1}{2}\mathbf{C}\mathbf{A}\mathbf{G}^{-1}\mathbf{D}^t & -\frac{1}{2}\mathbf{C}\mathbf{A}\mathbf{G}^{-1} & \mathbf{C} + \frac{1}{2}\mathbf{C}\mathbf{A}\mathbf{G}^{-1}\mathbf{A}^t\mathbf{C} \end{pmatrix} \tag{16}$$

where $\mathbf{D}' = 2\Big(\mathbf{B} - \mathbf{G} - \frac{1}{4}\mathbf{A}^t\mathbf{C}\mathbf{A}\Big)$. The duality transformations of the orbifold are then given by those elements of $O(d+16, 6; Z)$ commuting with $\mathcal{R}'$ and satisfying

$$\mathbf{T}^{-1}\mathbf{\Omega}^{-1}\mathbf{T} \in Z. \tag{17}$$

The condition (17) arising from the fact that the quantum numbers should transform by an integer-valued transformation, constrains the parameters of $\mathbf{\Omega}$ and thus breaks the duality group to a subgroup whose form depends on the choice of the discrete Wilson lines.

Having determined the duality group for the untwisted sector of orbifold compactification in the presence of Wilson lines, we now turn to discuss the symmetries of the twisted sectors. The twisted sectors are not sensitive to the background fields unless the associated twist leaves invariant a particular plane of the orbifold three complex planes. These twisted sectors are of particular importance as they contribute to the moduli-dependent threshold corrections to the gauge coupling constants [20-22].

Consider a k-twisted sector in a six-dimensional orbifold compactified heterotic string theory, in which the associated twist θ^k represented by $\mathcal{R}^k$ leaves a complex plane of the orbifold invariant. This sector will have quantum numbers satisfying

$$\mathbf{Q}^k\mathbf{n} = \mathbf{n}; \qquad \mathbf{Q}^{*k}\mathbf{m} = \mathbf{m}; \qquad \mathbf{M}^k\mathbf{l} = \mathbf{l}. \tag{18}$$

Let E_a, $a = 1, 2$ be a set of basis vectors for the directions of the orbifold invariant under the action of of the twist θ^k and $\mathcal{E}_\mu$, $\mu = 1, \cdots, d'$ be a basis for the invariant

$E_8 \times E_8$ directions. Clearly these invariant directions are expressed as some integral linear combinations of the orbifold lattice vectors e_i and the $E_8 \times E_8$ lattice vectors e_I. Then the twisted states will have a winding and momentum vectors L and P given by

$$L = \hat{n}^1 E_1 + \hat{n}^2 E_2, \quad P = \hat{m}_1 \tilde{E}_1 + \hat{m}_2 \tilde{E}_2 \tag{19}$$

where $\tilde{E}_1, \tilde{E}_2$ are certain linear combinations of the dual basis vectors e_i^* with $e_i^* \cdot e_j = \delta_{ij}$, and $\hat{n}^1, \hat{n}^2, \hat{m}_1, \hat{m}_2$ are integers. Note that $\tilde{E}_a$ are not necessarily orthogonal to E_b so we define $\tilde{E}_a \cdot E_b = \alpha_{ab}$. Represent the quantum numbers corresponding to the invariant directions by the matrices

$$\hat{\mathbf{n}} = \begin{pmatrix} \hat{n}^1 \\ \hat{n}^2 \end{pmatrix}; \quad \hat{\mathbf{m}} = \begin{pmatrix} \hat{m}_1 \\ \hat{m}_2 \end{pmatrix}; \quad \hat{\mathbf{l}} = \begin{pmatrix} \hat{l}_1 \\ \vdots \\ \hat{l}_{d'} \end{pmatrix} \tag{20}$$

and the background fields by the 2×2 matrices $\mathbf{G}_\perp$, $\mathbf{B}_\perp$ and the $d' \times 2$ matrix $\mathbf{A}_\perp$ where $\mathbf{G}_\perp$ and $\mathbf{B}_\perp$ are the metric and antisymmetric tensor of the invariant plane, $\mathbf{A}_\perp$ is the matrix representing the continuous Wilson lines A_a^μ. Clearly $\mathbf{G}_\perp$, $\mathbf{B}_\perp$ and $\mathbf{A}_\perp$ are constructed from the original 6×6 matrices $\mathbf{G}$ and $\mathbf{B}$ and from the 16×6 matrix $\mathbf{A}$ respectively and satisfy

$$\mathbf{Q}_\perp^t \mathbf{G}_\perp \mathbf{Q}_\perp = \mathbf{G}_\perp, \quad \mathbf{Q}_\perp^t \mathbf{B}_\perp \mathbf{Q}_\perp = \mathbf{B}_\perp, \quad \mathbf{M}_\perp \mathbf{A}_\perp = \mathbf{A}_\perp \mathbf{Q}_\perp, \tag{21}$$

where $\mathbf{Q}_\perp$ and $\mathbf{M}_\perp$ defines the action of the twist on the directions E_a and $\mathcal{E}_\mu$ respectively. Note also that $\mathbf{A}_\perp$ will have non-vanishing components only in the rotated directions of $\mathcal{E}_\mu$. Then in the lattice basis the twisted sector will have, in matrix notation, the following left and right moving momenta

$$\begin{aligned} \mathbf{P}_L =& \Big(\frac{\alpha \hat{\mathbf{m}}}{2} + (\mathbf{G}_\perp - \mathbf{B}_\perp - \frac{1}{4} \mathbf{A}_\perp^t \mathbf{C}_\perp \mathbf{A}_\perp) \hat{\mathbf{n}} - \frac{1}{2} \mathbf{A}_\perp^t \mathbf{C}_\perp \hat{\mathbf{l}},\ \hat{\mathbf{l}} + \mathbf{A}_\perp \hat{\mathbf{n}} \Big) = \Big(\mathbf{p}_L,\ \tilde{\mathbf{p}}_L \Big), \\ \mathbf{P}_R =& \Big(\frac{\alpha \hat{\mathbf{m}}}{2} - (\mathbf{G}_\perp + \mathbf{B}_\perp + \frac{1}{4} \mathbf{A}_\perp^t \mathbf{C}_\perp \mathbf{A}_\perp) \hat{\mathbf{n}} - \frac{1}{2} \mathbf{A}_\perp^t \mathbf{C}_\perp \hat{\mathbf{l}},\ \mathbf{0} \Big) = \Big(\mathbf{p}_R,\ \mathbf{0} \Big), \end{aligned} \tag{22}$$

where α is the matrix α_{ab} and $\mathbf{C}_\perp$ is a $d' \times d'$ representing the Cartan matrix of the directions $\mathcal{E}_\mu$. The world sheet energy and momentum of these twisted states

now takes the form

$$H = \frac{1}{2}\hat{\mathbf{u}}^t \boldsymbol{\Xi}_\perp \hat{\mathbf{u}}_\perp, \qquad P = \frac{1}{2}\hat{\mathbf{u}}^t \eta \hat{\mathbf{u}}, \quad \hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{n}} \\ \alpha\hat{\mathbf{m}} \\ \hat{\mathbf{l}} \end{pmatrix} \tag{23}$$

where $\eta_\perp$ and $\boldsymbol{\Xi}_\perp$ are defined by (5), with a two dimensional identity and the replacement of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{A}$ and $\mathbf{C}$ by $\mathbf{G}_\perp$, $\mathbf{B}_\perp$, $\mathbf{A}_\perp$ and $\mathbf{C}_\perp$ respectively. As an illustrative example, consider the orbifold $\mathbf{Z}_6 - II$, with the twist defined by $\theta = (2, 1, -3)/6$ and an $SU(6) \times SU(2)$ lattice.* The matrix Q defining the twist action on the quantum numbers is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{24}$$

Consider the θ^2-twisted sector which leave the the first $\mathbf{Z}_2$ complex plane invariant. The invariant direction of the lattice and its dual under the θ^2 action are given by

$$\begin{aligned} E_1 =& e_1 + e_3 + e_5, \quad E_2 = e_6, \\ \tilde{E}_1 =& e^1 - e^2 + e^3 - e^4 + e^5, \quad \tilde{E}_2 = e^6, \\ \text{with } \alpha_{ab} =& \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{25}$$

The matrices $\mathbf{G}_\perp$ and $\mathbf{B}_\perp$ which are defined in terms of the E_a can be easily extracted from $\mathbf{G}$ and $\mathbf{B}$, defining the background of the six-dimensional orbifold. Clearly the form of the Wilson lines $\mathbf{A}_\perp$ and the Cartan metric $\mathbf{C}_\perp$ will depend on the matrix $\mathbf{M}$, and its invariant directions.

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* the notation $(\zeta_1, \zeta_2, \zeta_3)$ is such that the action of θ in the complex basis is $(e^{2\pi i\zeta_1}, e^{2\pi i\zeta_2}, e^{2\pi i\zeta_3})$.

Next we study the duality symmetry of these twisted sectors. First let us consider the case where the invariant plane is a $\mathbf{Z}_2$-plane with continuous Wilson lines only. Moreover let us consider the situation where $\alpha = \mathbf{1_2}$, *i.e.*, the invariant directions lie entirely in a two dimensional sublattice of the orbifold six-dimensional lattice. We look for symmetries which leave both the spectrum of the twisted sector and $\hat{\mathbf{l}} + \mathbf{A}_\perp \hat{\mathbf{n}}$ invariant. This is sufficient to ensure the invariance of the threshold corrections. The most general form of the duality transformations with these requirements is given by

$$\mathbf{\Omega}_{\mathbf{Z}_2} = \begin{pmatrix} \mathbf{F} & \mathbf{0} & \mathbf{0} \\ \mathbf{F}^*\Big(\mathbf{J} - \frac{1}{2}\mathbf{V}_1^t\mathbf{C}_\perp\mathbf{V}_1\Big) & \mathbf{F}^* & -\mathbf{F}^*\mathbf{V}_1^t\mathbf{C}_\perp \\ \mathbf{V}_1 & \mathbf{0} & \mathbf{1}_{d'} \end{pmatrix} \tag{26}$$

where $\mathbf{F}$ is an $SL(2,Z)$ matrix satisfying $\mathbf{F}\mathbf{Q}_\perp = \mathbf{Q}_\perp\mathbf{F}$, $\mathbf{J}$ is any antisymmetric integer matrix and $\mathbf{V}_1$ is an integer matrix satisfying $\mathbf{V}_1\mathbf{Q}_\perp = \mathbf{M}_\perp\mathbf{V}_1$. Under the action of $\mathbf{\Omega}_{\mathbf{Z_2}}$ the transformations of the background field can be obtained from (8). However, a simpler method of obtaining the transformation law on the moduli was given in [9]. There one defines the projective coordinate $\mathbf{P} = \begin{pmatrix} -\mathbf{C}_\perp\mathbf{A}_\perp \\ \mathbf{D}_\perp \end{pmatrix}$, where $\mathbf{D}_\perp = 2\Big(\mathbf{B}_\perp - \mathbf{G}_\perp - \frac{1}{4}\mathbf{A}_\perp^t\mathbf{C}_\perp\mathbf{A}_\perp\Big)$. This coordinate transforms under the action of $\mathbf{\Omega}_{\mathbf{Z}_N}$ as

$$\begin{aligned} &\mathbf{P} \longrightarrow (\mathbf{X}_1\mathbf{P} + \mathbf{X}_2)(\mathbf{X}_3\mathbf{P} + \mathbf{X}_4)^{-1}, \\ &\mathbf{X}_1 = \begin{pmatrix} \mathbf{1}_{d'} & \mathbf{0} \\ \mathbf{V}_1^t & \mathbf{F}^t \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} -\mathbf{C}_\perp\mathbf{V}_1\mathbf{F}^{-1} \\ -(\mathbf{J} + \frac{1}{2}\mathbf{V}_1^t\mathbf{C}_\perp\mathbf{V}_1)\mathbf{F}^{-1} \end{pmatrix}, \\ &\mathbf{X}_3 = (\,\mathbf{0} \quad \mathbf{0}\,), \quad \mathbf{X}_4 = \mathbf{F}^{-1}. \end{aligned} \tag{27}$$

In particular, (27) gives for the Wilson lines,

$$\mathbf{A}_\perp \longrightarrow \mathbf{A}_\perp\mathbf{F} + \mathbf{V}_1. \tag{28}$$

Clearly, the duality group defined by (26) contains the U-duality as a subgroup.

This is the subgroup defined by

$$\mathbf{\Omega}_U = \begin{pmatrix} \mathbf{F} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{d'} \end{pmatrix} \tag{29}$$

with $det\mathbf{F} = 1$. Its action on the quantum numbers and the background fields is given by

$$\begin{aligned} \hat{\mathbf{n}} &\longrightarrow \mathbf{F}^{-1}\hat{\mathbf{n}}, \\ \hat{\mathbf{m}} &\longrightarrow \mathbf{F}^t\hat{\mathbf{m}}, \\ \hat{\mathbf{l}} &\longrightarrow \hat{\mathbf{l}}, \\ (\mathbf{G}_\perp \pm \mathbf{B}_\perp) &\longrightarrow \mathbf{F}^t(\mathbf{G}_\perp \pm \mathbf{B}_\perp)\mathbf{F}, \\ \mathbf{A}_\perp &\longrightarrow \mathbf{A}_\perp\mathbf{F}. \end{aligned} \tag{30}$$

Another symmetry is obtained by setting $\mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This symmetry, for the case where the Wilson line has two gauge indices and the rest vanishing, acts on the complex moduli defined in [9] by

$$T \longrightarrow \bar{T}, \quad U \longrightarrow \bar{U}, \quad B \longrightarrow \bar{C}, \quad C \longrightarrow \bar{B} \tag{31}$$

Another subgroup of (26) is the axionic shift symmetry which is given by

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{J} & \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{d'} \end{pmatrix} \tag{32}$$

This subgroup acts on the quantum numbers and moduli as

$$\begin{aligned} \hat{\mathbf{n}} &\longrightarrow \hat{\mathbf{n}} \\ \hat{\mathbf{m}} &\longrightarrow \hat{\mathbf{m}} - \mathbf{J}\hat{\mathbf{n}}, \\ \hat{\mathbf{l}} &\longrightarrow \hat{\mathbf{l}}, \\ \mathbf{G}_\perp &\longrightarrow \mathbf{G}_\perp, \\ \mathbf{B}_\perp &\longrightarrow \mathbf{B}_\perp - \frac{\mathbf{J}}{2}, \\ \mathbf{A}_\perp &\longrightarrow \mathbf{A}_\perp. \end{aligned} \tag{33}$$

This symmetry, is a subgroup of the T-duality symmetry. The other elements of the T duality do not leave $\hat{\mathbf{l}}+\mathbf{A}_\perp\hat{\mathbf{n}}$ invariant as it involves the interchange of windings and momenta as well as the mixing of $\mathbf{A}_\perp$ with $\mathbf{G}_\perp$ and $\mathbf{B}_\perp$. Therefore we conclude that the symmetries of the threshold corrections to the gauge coupling constants coming from a $\mathbf{Z}_2$ invariant plane are those which transform the continuous Wilson lines by an $SL(2,Z)$ rotation plus a shift. If we consider a $\mathbf{Z}_2$-plane with $\alpha \neq \mathbf{1}_2$, then the duality symmetry is given by (26) but with the additional condition

$$\begin{aligned} \alpha^{-1}\mathbf{F}^t\alpha &\in Z, \\ \alpha^{-1}\mathbf{V}_1^t\mathbf{C}_\perp &\in Z \\ \alpha^{-1}\Big(\mathbf{J}+\frac{1}{2}\mathbf{V}_1^t\mathbf{C}_\perp\mathbf{V}_1\Big) &\in Z. \end{aligned} \tag{34}$$

Such conditions arise from the fact the quantum numbers should transform as integers.

The above considerations can be extended to a $\mathbf{Z}_N$ plane with $N \neq 2$. Here the duality symmetry is given by

$$\mathbf{\Omega}_{\mathbf{Z}_N} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} & \mathbf{0} \\ \Big(\mathbf{J}-\frac{1}{2}\mathbf{V}_1^{'t}\mathbf{C}_\perp\mathbf{V}_1^{'}\Big) & \mathbf{1}_2 & -\mathbf{V}_1^{'t}\mathbf{C}_\perp \\ \mathbf{V}_1^{'} & \mathbf{0} & \mathbf{1}_{d'} \end{pmatrix} \tag{35}$$

with the condition $\mathbf{V}_1^{'}\mathbf{Q}_\perp = \mathbf{M}_\perp\mathbf{V}_1^{'}$ coming from the fact that $\mathbf{\Omega}_{\mathbf{Z}_N}$ has to commute with the $\mathbf{Z}_N$ twist. Also in the case when $\alpha \neq \mathbf{1}_2$, a further breaking of the symmetry occurs due to the conditions

$$\begin{aligned} \alpha^{-1}\mathbf{V}_1^{'t}\mathbf{C}_\perp &\in Z \\ \alpha^{-1}\Big(\mathbf{J}+\frac{1}{2}\mathbf{V}_1^{'t}\mathbf{C}_\perp\mathbf{V}_1^{'}\Big) &\in Z. \end{aligned} \tag{36}$$

If we now allow the presence of discrete Wilson lines $\mathbf{a}$, the twisted sectors with invariant planes have left and right moving momenta of the form (22) but with $\hat{\mathbf{n}}$,

$\hat{\mathbf{m}}$ and $\hat{\mathbf{l}}$ replaced by the $\hat{\mathbf{n}}'$, $\hat{\mathbf{m}}'$ and $\hat{\mathbf{l}}'$ where the latter quantum numbers are the independent entries of the solutions of

$$\mathbf{Q}^k \mathbf{n}' = \mathbf{n}'; \qquad \mathbf{Q}^{*k} \mathbf{m}' = \mathbf{m}'; \qquad \mathbf{M}^k \mathbf{l}' = \mathbf{l}'. \tag{37}$$

where $(\mathbf{n}', \mathbf{m}', \mathbf{l}')$ are the basis defined in (13). If we write

$$\hat{\mathbf{u}}' = \begin{pmatrix} \hat{\mathbf{n}}' \\ \hat{\mathbf{m}}' \\ \hat{\mathbf{l}}' \end{pmatrix} = \mathcal{T} \hat{\mathbf{u}} \tag{38}$$

where $\mathcal{T}$ depends on the choice of discrete Wilson lines of the orbifold model considered, then the duality symmetry of the threshold corrections in the presence of discrete Wilson lines are those symmetries obtained in the absence of discrete Wilson lines with the additional constraint

$$\mathcal{T}^{-1} \mathbf{\Omega}_{\mathbf{Z}_N} \mathcal{T} \in Z. \tag{39}$$

The condition (39) is the statement that the duality transformation should act on the quantum numbers $\hat{\mathbf{u}}$ by an integer-valued matrix. When $\mathbf{M} = \mathbf{I}$ (i.e. in the absence of continuous Wilson lines), the form of $\mathcal{T}$ for all $\mathbf{Z}_N$ orbifolds was derived in [10] .

In summary, we have studied the duality symmetry of orbifold compactification in the presence of a general background. We have derived the duality symmetries of the threshold corrections to the gauge coupling constants in the presence of continuous Wilson lines (Wilson line moduli), and demonstrated that these symmetries are broken to a subgroup if one includes discrete Wilson lines. The threshold corrections in the presence of Wilson lines could be of fundamental phenomenological importance in the study of the gauge couplings unification. In the absence of Wilson lines, the expectation values of the moduli which gives rise to the unification scale of the gauge coupling constant are not in agreement with those obtained

through the minimization of a possible non-perturbative superpotential. The dependence of the threshold corrections on the Wilson lines moduli gives more degrees of freedom which might resolve this discrepancy.

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